Selected Solutions for Chapter 3: Growth of Functions

Solution to Exercise 3.1-2

To show that $(n+a)^b = \Theta(n^b)$, we want to find constants $c_1, c_2, n_0 > 0$ such that $0 \le c_1 n^b \le (n+a)^b \le c_2 n^b$ for all $n \ge n_0$.

Note that

$$\begin{aligned} n + a &\le n + |a| \\ &\le 2n \qquad \text{when } |a| \le n \,, \end{aligned}$$

and

$$\begin{aligned} n + a &\ge n - |a| \\ &\ge \frac{1}{2}n \qquad \text{when } |a| \le \tfrac{1}{2}n \,. \end{aligned}$$

Thus, when $n \ge 2\,|a|$,

$$0 \le \frac{1}{2}n \le n + a \le 2n \,.$$

Since $b > 0$, the inequality still holds when all parts are raised to the power b:

$$0 \le \left(\frac{1}{2}n\right)^b \le (n+a)^b \le (2n)^b \,,$$

$$0 \le \left(\frac{1}{2}\right)^b n^b \le (n+a)^b \le 2^b n^b \,.$$

Thus, $c_1 = (1/2)^b$, $c_2 = 2^b$, and $n_0 = 2\,|a|$ satisfy the definition.

Solution to Exercise 3.1-3

Let the running time be $T(n)$. $T(n) \ge O(n^2)$ means that $T(n) \ge f(n)$ for some function $f(n)$ in the set $O(n^2)$. This statement holds for any running time $T(n)$, since the function $g(n) = 0$ for all n is in $O(n^2)$, and running times are always nonnegative. Thus, the statement tells us nothing about the running time.

Solution to Exercise 3.1-4

$2^{n+1} = O(2^n)$, but $2^{2n} \neq O(2^n)$.

To show that $2^{n+1} = O(2^n)$, we must find constants $c, n_0 > 0$ such that

$$0 \leq 2^{n+1} \leq c \cdot 2^n \text{ for all } n \geq n_0 .$$

Since $2^{n+1} = 2 \cdot 2^n$ for all n, we can satisfy the definition with $c = 2$ and $n_0 = 1$.

To show that $2^{2n} \neq O(2^n)$, assume there exist constants $c, n_0 > 0$ such that

$$0 \leq 2^{2n} \leq c \cdot 2^n \text{ for all } n \geq n_0 .$$

Then $2^{2n} = 2^n \cdot 2^n \leq c \cdot 2^n \Rightarrow 2^n \leq c$. But no constant is greater than all 2^n, and so the assumption leads to a contradiction.

Solution to Exercise 3.2-4

$\lceil \lg n \rceil!$ is not polynomially bounded, but $\lceil \lg \lg n \rceil!$ is.

Proving that a function $f(n)$ is polynomially bounded is equivalent to proving that $\lg(f(n)) = O(\lg n)$ for the following reasons.

- If f is polynomially bounded, then there exist constants c, k, n_0 such that for all $n \geq n_0$, $f(n) \leq cn^k$. Hence, $\lg(f(n)) \leq kc \lg n$, which, since c and k are constants, means that $\lg(f(n)) = O(\lg n)$.
- Similarly, if $\lg(f(n)) = O(\lg n)$, then f is polynomially bounded.

In the following proofs, we will make use of the following two facts:

1. $\lg(n!) = \Theta(n \lg n)$ (by equation (3.19)).
2. $\lceil \lg n \rceil = \Theta(\lg n)$, because
   - $\lceil \lg n \rceil \geq \lg n$
   - $\lceil \lg n \rceil < \lg n + 1 \leq 2 \lg n$ for all $n \geq 2$

$$\begin{aligned} \lg(\lceil \lg n \rceil!) &= \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil) \\ &= \Theta(\lg n \lg \lg n) \\ &= \omega(\lg n) . \end{aligned}$$

Therefore, $\lg(\lceil \lg n \rceil!) \neq O(\lg n)$, and so $\lceil \lg n \rceil!$ is not polynomially bounded.

$$\begin{aligned} \lg(\lceil \lg \lg n \rceil!) &= \Theta(\lceil \lg \lg n \rceil \lg \lceil \lg \lg n \rceil) \\ &= \Theta(\lg \lg n \lg \lg \lg n) \\ &= o((\lg \lg n)^2) \\ &= o(\lg^2(\lg n)) \\ &= o(\lg n) . \end{aligned}$$

The last step above follows from the property that any polylogarithmic function grows more slowly than any positive polynomial function, i.e., that for constants $a, b > 0$, we have $\lg^b n = o(n^a)$. Substitute $\lg n$ for n, 2 for b, and 1 for a, giving $\lg^2(\lg n) = o(\lg n)$.

Therefore, $\lg(\lceil \lg \lg n \rceil !) = O(\lg n)$, and so $\lceil \lg \lg n \rceil !$ is polynomially bounded.